

Sliding Bloom Filters

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Abstract

A Bloom filter is a method for reducing the space (memory) required for representing a set by allowing a small error probability. In this paper we consider a *Sliding Bloom Filter*: a data structure that, given a stream of elements, supports membership queries of the set of the last n elements (a sliding window), while allowing a small error probability. We formally define the data structure and its relevant parameters and analyze the time and memory requirements needed to achieve them. We give a low space construction that runs in $O(1)$ worst case time with high probability and provide an almost matching lower bound on the space that shows that our construction has the best possible space consumption up to an additive lower order term.

1 Introduction

Given a stream of elements, we consider the task of determining whether an element has appeared in the last n elements of the stream. To accomplish this task, one must maintain a representation of the last n elements at each step. One issue, is that the memory required to represent them might be too large and hence an approximation is used. We formally define this approximation and completely characterize the space and time complexity needed for the task.

In 1970 Bloom [Blo70] suggested an efficient data structure, known as the ‘*Bloom filter*’, for reducing the space required for representing a set S by allowing a small error probability on membership queries. The problem is also known as the approximate membership problem (however, we refer to any solution simply as a ‘Bloom filter’). A solution is allowed an error probability of ε for elements not in S (false positives), but no errors for members of S . In this paper, we consider the task of efficiently maintaining a Bloom filter of the last n elements (called ‘the sliding window’) of a stream of elements.

We define a (n, m, ε) -*sliding Bloom filter* as the task of maintaining a Bloom filter over the last n elements. The answer on these elements must always be ‘Yes’, the m elements that appear prior to them have no restrictions and for any other element the answers must be ‘Yes’ with probability at most ε . In case m is infinite, all elements prior to the current window have no restrictions. In this case we write for short (n, ε) -sliding Bloom filter.

The problem was studied in several communities and various solutions were suggested. In this work, we focus on a theoretical analysis of the problem and provide a rigorous analysis of the space and time needed for solving the task. We construct a sliding Bloom filter with $O(1)$ query and update time, where the is worst case with high probability over the stream (see the theorems in Section 1.2 for precise definitions) and has near optimal space consumption. We prove a matching space lower bound that is tight with our construction up to an additive lower order term. Our algorithms make use of a dynamic dictionary and given an implementation of one, it is relatively simple to complete the algorithm’s implementation.

A simple solution to the task is to partition the window into blocks of size m and for each block maintain its own Bloom filter. This results in maintaining $\lceil \frac{n}{m} + 1 \rceil$ Bloom filters. To determine if an element appeared

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or not we query all the Bloom filters and answer ‘Yes’ if any of them answered positively. There are immediate drawbacks of this solution, even assuming the Bloom filters are optimal in space and time:

- Slow query time: $\lceil \frac{n}{m} + 1 \rceil$ Bloom filter lookups.
- High error probability: since an error can occur on each block, to achieve an effective error probability of ε we need to set each Bloom filter to have error $\varepsilon' = \frac{\varepsilon m}{n+m}$, which means that the total space used has to grow (relative to a simple Bloom filter) by roughly $n \log \frac{n+m}{m}$ bits (see section 1.3).
- Sub-optimal space consumption for large m : the two first drawbacks are acute for small m , but when m is large, say $n = m$, then each block is large which results in a large portion of the memory being ‘wasted’ on old elements.

Sliding Bloom filters can be used in a wide range of applications and we discuss two settings where they are applicable and have been suggested. In one setting, Bloom filters are used to quickly determine whether an element is in a local cache [FCAB00], instead of querying the cache which may be slow. Since the cache has limited size, it usually stores the least recently used items (LRU policy). A sliding Bloom filter is used to represent the last n elements used and thus, maintain a representation of the cache’s contents at any point in time.

Another setting consists of the task of identifying duplicates in streams. In many cases, we consider the stream to be unbounded, which makes it impractical to store the entire data set and answer queries precisely and quickly. Instead, it may suffice to find duplicates over a sliding window while allowing some errors. In this case, a sliding Bloom filter (with m set to infinity) suffices and in fact, we completely characterize the space complexity needed for this problem.

1.1 Problem Definition

Given a stream of elements $\sigma = x_1, x_2, \dots$ from a finite universe U of size u , parameters n, m and ε , we want to approximately represent a sliding window of the n most recent elements of the stream. An algorithm A is given the elements of the stream one by one, and does not have access to previous elements that were not stored explicitly. Let $\sigma_t = x_1, \dots, x_t$ be the first t elements of the stream σ and let $\sigma_t(k) = x_{\max(0, t-k+1)}, \dots, x_t$ be the last k elements of the stream σ_t . At any step t the current window is $\sigma_t(n)$ and the m elements before them are $\sigma_{t-n}(m)$. If $m = \infty$ then define $\sigma_{t-n}(m) = x_1, \dots, x_{t-n}$. Denote $A(\sigma_t, x) \in \{\text{‘Yes’}, \text{‘No’}\}$ the result of the algorithm on input x given the stream σ_t . We call A a (n, m, ε) -sliding Bloom filter if for any $t \geq 1$ the following two conditions hold:

1. For any $x \in \sigma_t(n)$: $\Pr[A(x) = \text{‘Yes’}] = 1$
2. For any $x \notin \sigma_t(n+m)$: $\Pr[A(x) = \text{‘Yes’}] \leq \varepsilon$

where the probability is taken over the internal randomness of the algorithm A . Notice that for an element $x \in \sigma_{t-n}(m)$ the algorithm may answer arbitrarily (no restrictions). See Figure 1.

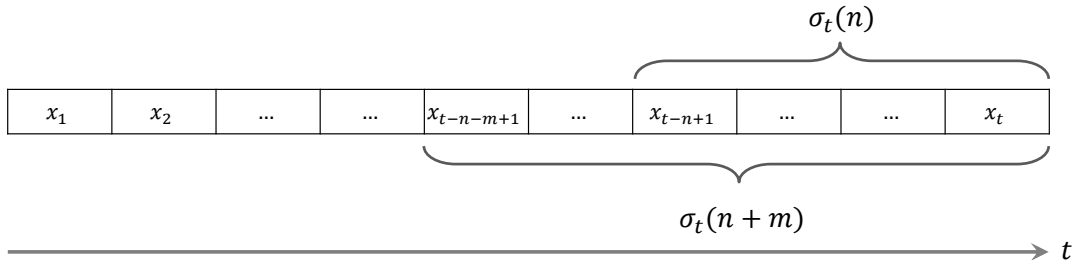


Figure 1: The sliding window of the last n and $n+m$ elements

An algorithm A for solving the problem is measured by its memory consumption, the time it takes to process each element and answer a query. We denote by $|A|$ the maximum number of bits used by A at

any step. The model we consider is the unit cost RAM model in which the elements are taken from a universe of size u , and each element can be stored in a single word of length $w = \log u$ bits. Any operation in the standard instruction set can be executed in constant time on w -bit operands. This includes addition, subtraction, bitwise Boolean operations, left and right bit shifts by an arbitrary number of positions, and multiplication. The unit cost RAM model is considered the standard model for the analysis of the efficiency of data structures.

1.2 Our Contributions

We provide tight upper and lower bounds to the (n, m, ε) -problem. Our first contribution is a construction of an efficient sliding Bloom filter: it has query time $O(1)$ worst case and update time $O(1)$ worst case with high probability. For $\varepsilon = o(1)$ the space consumption is near optimal.

Theorem 1.1. *For any $m > 0$, $\varepsilon = o(1)$ and sufficiently large n there exist an (n, m, ε) -sliding Bloom filter having the following space and time complexity on a unit cost RAM:*

1. *Query time is $O(1)$ worst case. For any polynomial $p(n)$ and sequence of at most $p(n)$ operation with probability at least $1 - 1/p(n)$ over the internal randomness of the data structure all insertions are performed in time $O(1)$ worst case.*
2. *Space consumption $(1 + o(1)) (n \log \frac{1}{\varepsilon} + \max \{n \log \log \frac{1}{\varepsilon}, n \log(\frac{n}{m})\})$*
3. *If $m = \infty$ then the space is $(1 + o(1)) (n \log \frac{1}{\varepsilon} + n \log \log \frac{1}{\varepsilon})$*

Our second contribution is a matching space lower bound. We prove that if $\varepsilon = o(1)$ then any sliding Bloom filter must use space that is within an additive low order term of the space of our construction, regardless of its running time. We assume that the sliding Bloom filter has the desired property that at *any* point the number of false positives is not too large (the ‘absolute false positive assumption’¹). In appendix A we state and prove a slightly weaker result without this assumption.

Theorem 1.2. *For any $m > 0$, $\varepsilon = o(1)$, sufficiently large n and an (n, m, ε) -sliding Bloom filter A if we assume that for any stream σ it holds that $\Pr[\exists i \leq 3n : |\{x \in U : A(\sigma_i, x) = \text{‘Yes’}\}| \geq 3\varepsilon u] \leq \frac{1}{2}$ then*

1. $|A| \geq n \log \frac{1}{\varepsilon} + \max \{n \log \log \frac{1}{\varepsilon}, n \log(\frac{n}{m})\} + O(n)$
2. *If $m = \infty$ then $|A| \geq n \log \frac{1}{\varepsilon} + n \log \log \frac{1}{\varepsilon} + O(n)$*

From Theorems 1.1 and 1.2 we conclude that making m larger than $n/\log \frac{1}{\varepsilon}$ does not make sense: one gets the same result for any value in $[n/\log \frac{1}{\varepsilon}, \infty)$. The lower bound is proved by an encoding argument which is a common way of showing lower bounds in this area (see for example [PSW13]). Specifically, the idea of the proof is to use A to encode a set S and a permutation π on the set corresponding to the order of the elements in the set. We consider the number of steps from the point an element is inserted to A to the first point where A answers ‘No’ on it, and we define λ to be the sum of n such lengths. If λ is large, then there is a point where A represents a large portion of S , which benefits in the encoding of S . If λ is small, then A can be used as an approximation of π , thus encoding π precisely requires a small amount of bits. In either case, the encoding must be larger than the entropy lower bound which yields a bound on the size of A . The optimal value of the trade-off between representing a larger set or representing a more accurate ordering is achieved by our construction. In this sense, our upper bound and lower bound match not only by ‘value’ but also by ‘structure’.

¹ The absolute false positive assumption is a very desirable property from a Sliding Bloom Filter and reasonable constructions enjoy it. We use it in the proof in section 3, and in appendix A we show how to get arbitrary close to the lower bound without it. An example of a Sliding Bloom Filter for which the assumption does not hold can be obtained by taking any (n, ε) -Sliding Bloom Filter and modify it such that it chooses a random index $k \in [1, n]$ and at step k of the stream it always answers ‘Yes’. This results in a $(n, \varepsilon + \frac{1}{n})$ -Sliding Bloom Filter in which there will *always* be some step at which the false positive rate is high (it is 1). One could add the assumption as a requirement to the definition of a Sliding Bloom Filter and it is not clear if a Sliding Bloom Filter can benefit from the absence of such a requirement.

1.3 Related Work and Background

The data structure for the approximate set membership (Bloom filter) as suggested by Bloom in 1970 [Blo70] is relatively simple: it consists of a bit array which is initiated to '0' and k random hash functions. Each element is mapped to k locations in the bit array using the hash functions. To insert an element set all k locations to 1. On lookup return 'Yes' if all k locations are 1. To achieve an error probability of ε for a set of size n Bloom showed that if $k = \log \frac{1}{\varepsilon}$ then the length of the bit array should be $\approx 1.44n \log \frac{1}{\varepsilon}$. Since its introduction the Bloom filter has been investigated extensively and many variants, implementations and applications of it have been suggested. A comprehensive survey (for its time) is Broder and Mitzenmacher [BM02].

A lot of attention was devoted for determining the exact space and time requirements of the approximate set membership problem. Carter et al. [CFG⁺78] proved an entropy lower bound of $n \log \frac{1}{\varepsilon}$, when the universe U is large. They also provided a reduction from approximate membership to *exact* membership, which we use in our construction. The retrieval problem associates additional data with each element of the set. In the static setting, where the elements are fixed and given in advance, Dietzfelbinger and Pagh propose a reduction from the retrieval problem to approximate membership [DP08]. Their construction gets arbitrarily close to the entropy lower bound. In the dynamic case, Lovett and Porat [LP10] proved that the entropy lower bound cannot be achieved for any *constant* error rate. They show a lower bound of $C(\varepsilon) \cdot n \log \frac{1}{\varepsilon}$ where $C(\varepsilon) > 1$ depends only on ε . Pagh, Segev and Wieder [PSW13] showed that if the size n is not known in advance then at least $(1 - o(1))n \log \frac{1}{\varepsilon} + \Omega(n \log \log n)$ bits of space must be used.

Pagh, Pagh and Rao [PPR05] used the reduction of Carter et al. to improve the original Bloom filter in several ways: Lookup time becomes $O(1)$ independent of ε , has succinct space consumption, uses explicit hash functions and supports deletion. In the dynamic, setting for a constant ε we do not know what is the leading term in the memory needed, however, for any sub-constant ε we know that the leading term is $n \log \frac{1}{\varepsilon}$: Arbritman, Naor and Segev present a solution which is optimal up to an additive lower order term (i.e., it is a succinct representation) [ANS10]. Thus, in this work we focus on sub-constant ε .

The model of sliding windows was first introduced by Datar et al. [DGIM02]. They consider maintaining an approximation of a statistic over a sliding window. They provide an efficient algorithm along with a matching lower bound.

Data structures similar to the sliding Bloom filters have been studied in the literature. The simple solution using $m = n$ consists of two large Bloom filters which are used alternatively. This method known as *double buffering* was proposed for classifying packets caches [CFL04]. Yoon [Yoo10] improved this method by using the two buffers simultaneously to increase the capacity of the data structure. Deng and Rafiei [DR06] introduced the Stable Bloom filter and used it to approximately detect duplicates in stream. Instead of a bit array they use an array of counters and to insert an element they set all associated counters to the maximal value. At each step, they randomly choose counters to decrease and hence older element have higher probability of being decreased and eventually evicted over time. Metwally et al. [MAA05] showed how to use Bloom filters to identify duplicates in click streams. They considered three models: Sliding Windows, Landmark Windows and Jumping Windows and discuss their relations. A comprehensive survey including many variations is given by Tarkoma et al. [TRL12]. However, as far as we can tell, no formal definition of a sliding Bloom filter as well as a rigorous analysis of its space and time complexity, appeared before.

2 The Construction of a Succinct Sliding Bloom Filter

Our algorithm applies the construction of an approximate membership data structure for a set S using the reduction from approximate membership to exact membership [CFG⁺78]. On an input x , we store $h(x)$ for some hash function h , in a dynamic dictionary and in addition some information on the last time where x appeared. We consider the stream to be divided to generations of size n/c each, where c is a parameter that will be optimized later. The first n/c elements are generation 1, the next n/c elements are generation 2 etc. The current window contains the last n elements and consists of at most $c + 1$ different generations. Therefore, at each step, we maintain a set S that represents the last $c + 1$ generations (that is, at most $n + n/c$ elements) and count the generations mod $c + 1$. In addition to storing $h(x)$, we associate $s = \log(c + 1)$ bits indicating the generation of x . Every n/c steps, we delete elements associated with the oldest generation. Finally, we adjust c to optimize the space consumption while requiring $n/c \leq m$.

In the rest of this section, we describe the algorithm in more detail. We first present the reduction from approximate to exact membership. We define a dynamic dictionary and the properties we need from it in order to implement our algorithm. Then, we describe the algorithm in two stages, using any dictionary as a black box. The memory consumption is merely the memory of the dictionary and therefore we use one with succinct representation. At first, the running time will not be optimal and depend on c (which is not a constant), even if we use optimal dictionaries. Then, we describe how to eliminate the dependency on c and well as deamortizing the algorithm, making the running time constant for each operation. This includes augmenting the dictionary, and thus it can no longer be treated as a black box. We prove correctness and analyze the resulting memory consumption and running time.

2.1 A Reduction to Exact Membership

We want to represent a set S of size n and support membership queries in the following manner: For a query on $x \in S$ we answer ‘Yes’ and for $x \notin S$ we answer ‘Yes’ with probability at most ε . Choose a hash function $h \in \mathcal{H}$ from a universal family of hash functions mapping $U \rightarrow [n/\varepsilon]$. Then for any S of size at most n it holds that for any $x \in U$:

$$\Pr_h[h(x) \in h(S)] \leq \sum_{y \in S} \Pr_h[h(x) = h(y)] \leq n \frac{\varepsilon}{n} = \varepsilon$$

where the first inequality comes from a union bound and the second from the definition of a universal hash family. This implies that storing $h(S)$ suffices for solving the approximate membership problem. To store $h(S)$ we use an exact dictionary \mathcal{D} , which supports **insert** (including associated data), **delete** and **update** procedures (the update procedure can be simulated by a delete followed by an insert). While most dictionaries support these basic procedures, we require \mathcal{D} to additionally support the ability of *scanning*. We further discuss these properties.

2.2 Succinct Dynamic Dictionary

The information-theoretic lower bound on the minimum number of bits needed to represent a set S of size n out of M different elements is $\mathcal{B} = \mathcal{B}(M, n) = \left\lceil \log \binom{M}{n} \right\rceil = n \log M - n \log n + O(n)$. A succinct representation is one that uses $(1 + o(1))\mathcal{B}$ bits [Dem07]. A significant amount of work was devoted for constructing dynamic dictionaries over the years and most of them are appropriate for our construction. Some have good theoretical results and some emphasize the actual implementation. In order for the reduction to compete with the Bloom filter construction (in terms of memory consumption) we must use a dynamic dictionary with succinct representation. There are several different definitions in the literature for a *dynamic* dictionary. A static dictionary is a data structure storing a finite subset of a universe U , supporting only the **member** operation. In this paper, we refer to a dynamic dictionary where only an upper bound n on the size of S is given in advance and it supports the procedures **member**, **insert** and **delete**. The memory of the dictionary is measured with respect to the bound n .

In addition to storing $h(S)$, we assume \mathcal{D} supports associating data with each element. Specifically, we want to store s -bits of data with each element, where s is fixed and known in advance. Finally, we assume the dictionary supports *scanning*, that is, the ability to go over the associated data of all elements of the dictionary, and delete the element if needed. Using the scanning process, we scan the generations stored in the dictionary and delete elements of specific generations.

Several dynamic dictionaries can be used in our construction of a Sliding Bloom Filter. The running time and space consumption are directly inherited from the dictionary, making it an important choice. We use the construction of [ANS10] (but other alternative are possible). It supports **insert** and **delete** in $O(1)$ worst case with high probability while having a succinct representation. Implicitly in their work, they support associating any fixed number of bits and scanning. When s -bits of data is associated with each $x \in S$, the representation lower bound becomes $\mathcal{B} + ns$ bits. For concreteness, the memory consumption of their dictionary is $(1 + o(1))(\mathcal{B} + ns)$, where the $o(1)$ hides the expression $\frac{\log \log n}{\log^{1/3} n}$.

2.3 Algorithm with Dependency on ε

Initiate a dynamic dictionary \mathcal{D} of size $n' = n(1 + \frac{1}{c})$ as described above. Let $\mathcal{H} = \{h : U \rightarrow [n'/\varepsilon]\}$ be a family of universal hash functions and pick $h \in \mathcal{H}$ at random. At each step maintain a counter ℓ indicating the current generation and a counter i indicating the current element in the generation. At every step i is increased and every n/c steps i is reset back to 0 and ℓ is increased mod $c + 1$.

To insert an element x check if $h(x)$ exists in \mathcal{D} . If not then insert $\langle h(x), \ell \rangle$ (insert $h(x)$ associated with ℓ) into \mathcal{D} . If $h(x)$ is in \mathcal{D} , then update the associated data of $h(x)$ to ℓ . Finally, update the counters i and ℓ . If ℓ has increased (which happens every n/c steps) then *scan* \mathcal{D} and delete all elements with associated data equal to the new value of ℓ .

To query the data structure on an element x , return whether $h(x)$ is in \mathcal{D} . See appendix B for pseudo-code of the insert and lookup procedures (See Algorithm 1).

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Insert( $x$ ) :
1: if  $h(x)$  is a member of  $\mathcal{D}$  then
2:   update  $h(x)$  to have data  $\ell$ 
3: else
4:   insert  $\langle h(x), \ell \rangle$  into  $\mathcal{D}$ 
5: end if
6: maintain counters  $i$  and  $\ell$ 
7: if the value of  $\ell$  has changed then
8:   scan  $\mathcal{D}$  and delete elements of generation  $\ell$ 
9: end if

Lookup( $x$ ) :
1: procedure MEMBER( $x$ )
2:   if  $h(x)$  is a member of  $\mathcal{D}$  then
3:     return ‘Yes’
4:   else
5:     return ‘No’
6:   end if
7: end procedure

```

Algorithm 1: Pseudo-code of the Insert and Lookup procedures

Correctness: We first notice that \mathcal{D} is used correctly and never represents a set of size larger than n' . In each step we either insert an element to generation ℓ or move an existing element to generation ℓ . In any case, each generation consists of at most n/c elements in \mathcal{D} . Each n/c we evict a whole generation, assuring no more than $c + 1$ generations are present in the dictionary at once. Thus, at most n' are represented at any given step.

Next we prove that for any time t the two conditions hold. The first condition follows directly from the algorithm. Assume $h(x)$ is inserted with associated generation $\ell = j$. Notice that its associated generation can only increase. $h(x)$ will be deleted only when ℓ completes a full cycle and its value is j again, which takes at least n steps. Thus, for any $x \in \sigma_t(n)$, $h(x)$ is in \mathcal{D} and the algorithm will always answer ‘Yes’.

For the second condition assume that $x \notin \sigma_t(n + m)$ and notice that $n + m$ is exactly $c + 1$ generations. Assume w.l.o.g. that $S = \{y_1, \dots, y_{n'}\}$ (S could have less than n' elements) is the set of elements represented in \mathcal{D} at time t . Then $\Pr[h(x) = y_i] = \frac{\varepsilon}{n'}$ for all $i \in [n']$. Therefore, using a union bound we get that the total false positive probability is

$$\Pr[A(x) = \text{‘Yes’}] = \Pr[h(x) \in h(S)] \leq \sum_{i=1}^{n'} \Pr[h(x) = y_i] \leq \varepsilon$$

Memory consumption: The bulk of memory is used for storing \mathcal{D} . In addition, we need to store two counters i and ℓ and the hash function h , which together take $O(\log n)$ bits. \mathcal{D} stores n' elements out of

$M = \lceil n'/\varepsilon \rceil$ while associating each with $s = \log c$ bits. Using the dictionary of [ANS10] yields a total space of

$$(1 + o(1))(\mathcal{B}(\frac{n'}{\varepsilon}, n') + n's) = (1 + o(1)) \cdot n \left(1 + \frac{1}{c}\right) \left(\log \frac{1}{\varepsilon} + \log c + 1\right)$$

We minimize this expression, as a function of c , and get that the minimum is at the solution to $c - \log c = \log \frac{1}{\varepsilon} - 1$. An approximate solution is $c = \log \frac{1}{\varepsilon}$ and requiring that $n/c \leq m$ yields that $c = \max \{\log \frac{1}{\varepsilon}, m/n\}$ and the total space is

$$(1 + o(1)) \left(n \log \frac{1}{\varepsilon} + n \cdot \max \left\{ \log \log \frac{1}{\varepsilon}, \log(n/m) \right\} \right)$$

if $m = \infty$ (or $m \geq \log \frac{1}{\varepsilon}$) then $c = \log \frac{1}{\varepsilon}$ and we get

$$(1 + o(1)) \left(n \log \frac{1}{\varepsilon} + n \log \log \frac{1}{\varepsilon} \right)$$

as required. As mentioned, the $o(1)$ hides the term $\frac{\log \log n}{\log^{1/3} n}$, therefore if $\varepsilon = 2^{-O(\frac{\log \log n}{\log^{1/3} n})}$ the space consumption can be written as $n \log \frac{1}{\varepsilon} + n \log \log \frac{1}{\varepsilon} + O(n)$ which matches the lower bound up to the $O(n)$ term.

Running time: Assume that \mathcal{D} supports $O(1)$ running time worst case for all procedures. The lookup procedure performs a single query to \mathcal{D} and hence always runs in $O(1)$. In the insert procedure, every n/c steps, the value of ℓ is updated and we scan all elements in \mathcal{D} deleting old elements. For any other step, the running time is $O(1)$. Therefore, the total running time for n/c steps is $O(n')$, which is $O(c)$ amortized running time. If $m \geq \log \frac{1}{\varepsilon}$ then $c = \log \frac{1}{\varepsilon}$ and the running time is $(\log \frac{1}{\varepsilon})$, otherwise it is $O(\frac{n}{m})$, which in both cases is not constant. We now show how to eliminate the large step, making the running time $O(1)$ worst case. Using the dictionary of [ANS10] we get that the total running time including the dictionary's operations is $O(1)$ worst case with high probability (over internal randomness of the dictionary).

2.4 Eliminating the Large Step and Reducing the Time

We now modify the algorithm to eliminate the large step and reduce the time such that we get rid of the dependency on ε . These modifications require some additional properties from the dictionary. Later, for concreteness, we describe how to modify the dictionary of [ANS10] to support these properties.

We need the scanning process to support running in multiple steps while allowing other operation running in parallel. The scanning procedure should be able to save its state, then allow other operations to run and finally resume its state and continue the scanning process. In case elements have been added, moved or deleted the scanning process should continue scanning all elements nevertheless.

In general, modifying a dictionary to support this is hard as the dictionary might implicitly represents many elements using the same memory space. However, it can be implemented easily assuming each element has a unique memory space in which it is (implicitly) represented, called a 'cell'. An insert or delete procedure may modify a constant number of cells. Elements of cells which were accessed are called the accessed elements. We assume the cells have some order in which we can scan them and save an index indicating the state of the scanning process using $o(n)$ bits of memory (actually it is $O(\log n)$). We assume that given a cell, we can figure out the associated data with the element of the cell and delete the element of this cell from the dictionary.

Using these assumptions, we can eliminate the large step by scanning and deleting old generations over a process of many steps. An element is considered old if it is more than $c + 1$ generations older than the current generation counter, ℓ . When scanning an element we examine its generation and delete the element if it's old. Instead of scanning all the n' elements in one step, we scan two elements at each step and save the scanning index. Thus, after $n'/2$ steps we scan all n' elements of the dictionary. A problem that occurs, is that the dictionary can change during the process, which may result in the scanning missing cells. To solve this, we modify the dictionary's insert and delete procedures to check whether any accessed element needs to be deleted. For example, an insert procedure may move an element from one cell to another, which was already scanned. Thus, before moving or changing a cell we scan it. This way, each element is scanned either by the scanning process or by an insert or delete procedure.

This implies that an old element might be deleted only after $n'/2$ steps, which means that there could be up to $2c + 1$ different generations stored in \mathcal{D} at the same time. We need to extend the range of the counter ℓ to loop between 0 and $2c + 1$ (instead of between 0 and $c + 1$), in order to represent all generations. Only the $c + 1$ recent generations are considered active and the rest slated to be deleted. We change the **Lookup** procedure to return ‘Yes’ on input x only if $h(x)$ exists in \mathcal{D} **and** its associated generation is active. These modifications have negligible effect on the memory consumption and running time and preserve the correctness of the algorithm.

We discuss implementing these requirements in the construction of [ANS10] (see pseudo-code on page 9 of their paper). Their hashing scheme is based on two-level hashing, the first level consists of an array T_0 of bins of size d and the second level consists of cuckoo hashing which includes two arrays, T_1 and T_2 and a queue, Q . The cells are the d cells in each bin of T_0 , the cells of T_1 , T_2 and Q . Each element is implicitly stored in a unique cell in one of the components. Scanning the cells is achieved by going over the cells of each component and saving an index of the current component and cell within the component. The delete procedure is simple and does not involve moving cells. The insert procedure is more involved and may move cells from one component to another, e.g. a cell from Q might be moved to T_0 . Since the running time is constant, so is the number of accessed elements. The procedure can be easily modified such that before a cell is accessed it is first scanned, and deleted if old. After these modification, the dictionary of [ANS10], supports all needed requirements and hence this completes our construction of a (n, ε) -Sliding Bloom Filter.

3 A Tight Space Lower Bound

In this section we present a matching space lower bound to our construction. We restate and prove Theorem 1.2.

Theorem 1.2 (Restated). *For any $m > 0$, $\varepsilon = o(1)$, sufficiently large n and an (n, m, ε) -sliding Bloom filter A if we assume that for any stream σ it holds that $\Pr[\exists i \leq 3n : |\{x \in U : A(\sigma_i, x) = \text{‘Yes’}\}| \geq 3\varepsilon u] \leq \frac{1}{2}$ then*

1. $|A| \geq n \log \frac{1}{\varepsilon} + \max \{n \log \log \frac{1}{\varepsilon}, n \log(\frac{n}{m})\} + O(n)$
2. *If $m = \infty$ then $|A| \geq n \log \frac{1}{\varepsilon} + n \log \log \frac{1}{\varepsilon} + O(n)$*

Proof. Let A be an algorithm breaking the space requirement in the statement of the theorem. The main idea of the proof is to use A to encode and decode a set $S \subset U$ and a permutation π on the set (i.e. an ordered set). Giving S to A as a stream, ordered by π , encodes an approximation of S and π . The decoder uses the encoding of A to compute all the elements of U for which A answers ‘Yes’. Then we encode only the elements of S from within this set. To decode π , the decoder checks how many elements are needed to be added to the stream in order for A to “release” each of the elements in S (that is, to answer ‘No’ on it). Then we encode only the difference between the location i and the location it has been released.

Denote by A_r the algorithm with fixed random string r and let $\mu_{A_r}(\sigma) = \{x : A_r(\sigma, x) = \text{‘Yes’}\}$. We show that w.l.o.g. we can consider A to be deterministic. Since $\Pr_r[\exists i \leq 3n : |\mu_{A_r}(\sigma_i)| \geq \varepsilon u] \leq \frac{1}{2}$ we fix r^* to be such that for all $1 \leq i \leq 3n$ it holds that $|\mu_{A_{r^*}}(\sigma_i)| \leq \varepsilon u$.

Notice that r^* need not be explicitly specified in the encoding since the decoder can compute it using only the algorithm of A . From now on, we assume that A is deterministic (and remove the A_r notation) and assume that for any σ of length $2n$ we have that $\mu_A(\sigma) \leq 3\varepsilon u$. We now make an important definition:

$$\ell(\sigma, x) = \min_k \{ \arg \min \{ \exists y_1, \dots, y_k \in U : A(\sigma y_1 \dots y_k, x) = 0 \}, n, m \}$$

$\ell(\sigma, x)$ is the minimum number of elements needed to be added to σ such that A answers ‘No’ on x . Notice that $\ell_\sigma(\cdot)$ can be computed for any set S given the representation of $A(\sigma)$.

We encode any set S of size $2n$ and a permutation $\pi : [2n] \rightarrow [2n]$ using A . After encoding S we compare the encoding length to the entropy lower bound of $\mathcal{B}(u, 2n) + \log((2n)!)$. Consider applying π on (some canonical order of) the elements of S and let x_1, \dots, x_{2n} be the resulting elements of S ordered by π . For any $i > 2n$ let $x_i = x_{i-2n}$, then for any $k \geq 1$ define the sequence $\sigma_k = x_1, \dots, x_k$. Let $\phi(\sigma_k) = \mu(\sigma_k) \cap S$ and define

$$\Delta(\sigma_k, i) = \ell(\sigma_k, x_i) + (k - n) - i$$

Notice that, given $A(\sigma_k)$, $\Delta(\sigma_k, i)$, k and n one can compute the position i of the element x_i . Define

$$\lambda_k = \sum_{i=k-n+1}^k \Delta(\sigma_k, i), \text{ and } \lambda = \max_{n \leq k \leq 3n} \lambda_k$$

Notice that if $m \geq n$ (or $m = \infty$) then $0 \leq \lambda \leq n^2$, otherwise $0 \leq \lambda \leq nm$

Lemma 3.1. *There exist a k' , $n \leq k' \leq 4n$ such that $|\phi(\sigma_{k'})| \geq n + \frac{\lambda}{n}$.*

Proof. Let k^* be such that $\lambda = \lambda_{k^*}$ and consider σ_{k^*} . By an averaging argument, it is enough to show that

$$\sum_{j=k^*}^{k^*+n-1} |\phi(\sigma_j)| \geq n^2 + \lambda.$$

For any $k^* - n + 1 \leq i \leq k^*$ we know that $x_i \in \sigma_{k^*}(n)$ and by the definition of $\ell(\sigma_{k^*}, x_i)$ we know that $x_i \in \phi(\sigma_k^*), \dots, \phi(\sigma_{k^*+\ell(\sigma_{k^*}, x_i)})$. For any $k^* + 1 \leq i \leq k^* + n - 1$ we know that $x_i \in \sigma_{k^*+n-1}(n)$ and therefore $x_i \in \phi(\sigma_i), \dots, \phi(\sigma_{k^*+n-1})$. Instead of summing over $\phi(\sigma_j)$ we sum over x_i and count the number of $\phi(\sigma_j)$ such that $x_i \in \phi(\sigma_j)$:

$$\begin{aligned} \sum_{j=k^*+1}^{k^*+n} |\phi(\sigma_j)| &\geq \sum_{i=k^*-n+1}^{k^*} (\ell(\sigma_{k^*}, x_i) + 1) + \sum_{i=k^*+1}^{k^*+n-1} (k^* + n - i) \\ &= \sum_{i=k^*-n+1}^{k^*} (\ell(\sigma_{k^*}, x_i) + k^* - n - i) + \sum_{i=k^*-n+1}^{k^*} (i + n - k^* + 1) + \frac{n(n-1)}{2} \\ &= \sum_{i=k^*-n+1}^{k^*} \Delta(\sigma_{k^*}, i) + \frac{n(n+3)}{2} + \frac{n(n-1)}{2} \\ &\geq \lambda_{k^*} + n^2 = n^2 + \lambda \end{aligned}$$

□

Fix k' from Lemma 3.1. Since λ is the maximum of all the λ_k 's, we know also that $\lambda_{k'} \leq \lambda$. We include the memory representation of $A(\sigma_{k'})$ in the encoding. The decoder uses this to compute the set $\mu(\sigma_{k'})$, which by the absolute false positive definition we know that $|\mu(\sigma_{k'})| \leq 3\epsilon u$. By Lemma 3.1, we know that $|\phi(\sigma_{k'})| \geq n + \frac{\lambda}{n}$, therefore we use $\mathcal{B}(3\epsilon u, n + \frac{\lambda}{n})$ bits to encode $n + \frac{\lambda}{n}$ elements of S out of them. The remaining $n - \frac{\lambda}{n}$ elements are encoded explicitly using $\mathcal{B}(u, n - \frac{\lambda}{n})$ bits. This completes the encoding of S .

To encode π we need the decoder to be able to extract i for each x_i . For any $x_i \in \sigma_{k'}(n)$ the decoder uses $A(\sigma_{k'})$ and computes $\ell(\sigma_{k'}, x_i)$. Now, in order for the decoder to exactly decode i we need to encode all the $\Delta(\sigma_{k'}, i)$'s. Since $\sum_{i=k'-n+1}^{k'} \Delta(\sigma_{k'}, i) \leq \lambda$ we can encode all the $\Delta(\sigma_{k'}, i)$'s using $\log \binom{n+\lambda}{n}$ bits (balls and urns method), and the remaining elements' positions will be explicitly encoded using $n \log n$ bits. Denote by $|A|$ the number of bits used by the algorithm A . Comparing the encoding length to the entropy lower bound we get

$$|A| + \log \binom{3\epsilon u}{n + \frac{\lambda}{n}} + \log \binom{u}{n - \frac{\lambda}{n}} + \log \binom{n+\lambda}{n} + n \log n \geq \log \binom{u}{2n} + \log((2n)!)$$

and therefore

$$|A| \geq (n + \frac{\lambda}{n}) \log \frac{1}{\epsilon} + (n + \frac{\lambda}{n}) \log n + (n - \frac{\lambda}{n}) \log (n - \frac{\lambda}{n}) - n \log (\lambda + n) + O(n)$$

Consider two possible cases for λ . If $\lambda \leq 0.9n^2$ then we get

$$|A| \geq (n + \frac{\lambda}{n}) \log \frac{1}{\epsilon} + 2n \log n - n \log (\lambda + n) + O(n)$$

The minimum of this expression, as a function of λ , is achieved at $\lambda = \frac{n^2}{\log \frac{1}{\varepsilon}} - n$. If $m \geq \frac{n}{\log \frac{1}{\varepsilon}} - 1$ then the minimum can be achieved and we get that

$$|A| \geq n \log \frac{1}{\varepsilon} + n \log \log \frac{1}{\varepsilon} + O(n).$$

Otherwise, if $m < \frac{n}{\log \frac{1}{\varepsilon}} - 1$ then $\lambda \leq mn \leq \frac{n^2}{\log \frac{1}{\varepsilon}} - n$ and minimum value will be achieved at $\lambda = nm$ which yields the required lower bound:

$$|A| \geq n \log \frac{1}{\varepsilon} + n \log \frac{m}{n} + O(n).$$

If $0.9n^2 < \lambda \leq n^2$ then $m \geq 0.9n \geq \frac{n}{\log \frac{1}{\varepsilon}}$. Thus, we get that

$$|A| \geq \left(n + \frac{\lambda}{n}\right) \log \frac{1}{\varepsilon} - \left(n - \frac{\lambda}{n}\right) \log n + \left(n - \frac{\lambda}{n}\right) \log \left(n - \frac{\lambda}{n}\right) + O(n)$$

the minimum of this expression, as a function of λ between given range is achieved at $\lambda = 0.9n^2$ which yields

$$|A| \geq n \log \frac{1}{\varepsilon} + n \log \log \frac{1}{\varepsilon} + O(n)$$

as required. □

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References

- [ANS10] Yuriy Arbitman, Moni Naor, and Gil Segev. Backyard cuckoo hashing: Constant worst-case operations with a succinct representation. In *Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on*, pages 787–796. IEEE, 2010.
- [Blo70] Burton H. Bloom. Space/time trade-offs in hash coding with allowable errors. *Communications of the ACM*, 13(7):422–426, 1970.
- [BM02] Andrei Broder and Michael Mitzenmacher. Network applications of bloom filters: A survey. In *Internet Mathematics*, pages 636–646, 2002.
- [CFG⁺78] Larry Carter, Robert Floyd, John Gill, George Markowsky, and Mark Wegman. Exact and approximate membership testers. In *Proceedings of the tenth annual ACM Symposium on Theory of Computing*, pages 59–65. ACM, 1978.
- [CFL04] Francis Chang, Wu-chang Feng, and Kang Li. Approximate caches for packet classification. In *INFOCOM 2004. Twenty-third Annual Joint Conference of the IEEE Computer and Communications Societies*, volume 4, pages 2196–2207. IEEE, 2004.
- [Dem07] Erik Demaine. Lecture notes for the course "Advanced data structures". Available at <http://courses.csail.mit.edu/6.851/spring07/scribe/lec21.pdf>, 2007.
- [DGIM02] Mayur Datar, Aristides Gionis, Piotr Indyk, and Rajeev Motwani. Maintaining stream statistics over sliding windows. *SIAM Journal on Computing*, 31(6):1794–1813, 2002.
- [DP08] Martin Dietzfelbinger and Rasmus Pagh. Succinct data structures for retrieval and approximate membership. *Automata, Languages and Programming*, pages 385–396, 2008.
- [DR06] Fan Deng and Davood Rafiei. Approximately detecting duplicates for streaming data using stable bloom filters. In *Proceedings of the 2006 ACM SIGMOD international conference on Management of data*, pages 25–36. ACM, 2006.

- [FCAB00] Li Fan, Pei Cao, Jussara Almeida, and Andrei Z. Broder. Summary cache: a scalable wide-area web cache sharing protocol. *IEEE/ACM Trans. Netw.*, 8(3):281–293, June 2000.
- [LP10] Shachar Lovett and Ely Porat. A lower bound for dynamic approximate membership data structures. In *Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on*, pages 797–804. IEEE, 2010.
- [MAA05] Ahmed Metwally, Divyakant Agrawal, and Amr El Abbadi. Duplicate detection in click streams. In *Proceedings of the 14th international conference on World Wide Web*, pages 12–21. ACM, 2005.
- [PPR05] Anna Pagh, Rasmus Pagh, and S Srinivasa Rao. An optimal bloom filter replacement. In *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 823–829. Society for Industrial and Applied Mathematics, 2005.
- [PSW13] Rasmus Pagh, Gil Segev, and Udi Wieder. How to approximate a set without knowing its size in advance. *arXiv preprint arXiv:1304.1188*, 2013.
- [TRL12] Sasu Tarkoma, Christian Esteve Rothenberg, and Eemil Lagerspetz. Theory and practice of bloom filters for distributed systems. *Communications Surveys & Tutorials, IEEE*, 14(1):131–155, 2012.
- [Yoo10] MyungKeun Yoon. Aging bloom filter with two active buffers for dynamic sets. *Knowledge and Data Engineering, IEEE Transactions on*, 22(1):134–138, 2010.

A Removing the Absolute False Positive Assumption

We describe how to eliminate the absolute false positive assumption in the lower bound, which yields a slightly weaker results.

Theorem A.1. *For any $m > 0$, $\delta > 0$, $\varepsilon = o(1)$ and sufficiently large n if A is an (n, m, ε) -sliding Bloom filter A then*

1. $|A| \geq n \log \frac{1}{\varepsilon} + (1 - \delta) \max \left\{ n \log \log \frac{1}{\varepsilon}, n \log \left(\frac{n}{m} \right) \right\} + O(n)$
2. *If $m = \infty$ then $|A| \geq n \log \frac{1}{\varepsilon} + (1 - \delta)n \log \log \frac{1}{\varepsilon} + O(n)$*

Proof. (sketch) The proof of this theorem is similar to the proof of Theorem 1.2 and we describe the needed modification. The absolute false positive assumption assures us that with high probability for *any* $i \leq 3n$ we have that $|\mu(\sigma_i)| \leq 3\varepsilon u$, and hence we fix some r^* on which it holds. Since we don’t use the assumption, we show that there exist some r^* which for *most* i ’s the condition holds.

Claim A.2. *There exist some r^* such that for any σ of length $2n$: $|\{i : |\mu_{A_{r^*}}(\sigma_i)| \geq 11\varepsilon u\}| \leq \frac{n}{5}$*

Proof. By the requirement of a sliding Bloom filter we get that for any i , $\mathbb{E}_r[|\mu_{A_r}(\sigma_i)|] \leq 2n + \varepsilon u$. Then by Markov’s inequality we get $\Pr_r[|\mu_{A_r}(\sigma_i)| > 10(2n + \varepsilon u)] \leq \frac{1}{10}$. Assuming $u \gg n$ we get that $\Pr_r[|\mu_{A_r}(\sigma_i)| > 11\varepsilon u] < \frac{1}{10}$. Define $I = \mathbb{E}_r[|\{i \leq 2n : |\mu_{A_r}(\sigma_i)| \geq 11\varepsilon u\}|]$ to be the number of ‘bad’ indexes i.e. they have too many false positives. Then by linearity of expectation we get that $I = 2n \Pr_r[|\mu_{A_r}(\sigma_i)| > 11\varepsilon u] \leq \frac{n}{5}$. Thus, there must be a specific r^* such that $|\{i : |\mu_{A_{r^*}}(\sigma_i)| \geq 11\varepsilon u\}| \leq \frac{n}{5}$. \square

Fix k from lemma 3.1 of the proof. The problem is that the Lemma 3.1 doesn’t assure that k isn’t in the set I of ‘bad’ indexes, where the false positive rate is too high. We need to modify the lemma to compute the average over all indexes that are not in I . This results in a weaker version of the lemma stating that $|\phi(\sigma_k)| \geq \frac{4}{5} \left(n + \frac{\lambda}{n} \right)$. Of course, the choice of $\frac{4}{5}$ is arbitrary and could easily be changed to $(1 - \delta)$ for any $\delta > 0$ and resulting in a larger constant in the $O(n)$ term. This increases the number of bits needed for the encoding and results in the lower bound:

$$|A| \geq n \log \frac{1}{\varepsilon} + (1 - \delta)n \log \log \frac{1}{\varepsilon} + O(n)$$

\square